Lecture Notes
Analysis II

For Engineering Students

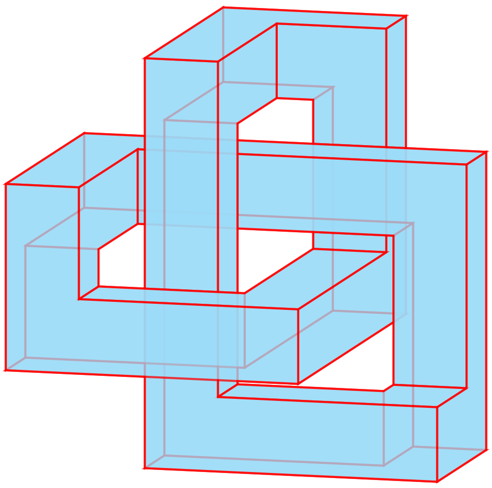
Spring Semester 2025

Contents

Chapter 1

The Euclidean space $\mathbb{R}^n$

In Analysis 1 you have learned the fundamental concepts of differential and integral calculus of real-valued functions in one real variable, known as *Single Variable Calculus*. However, real-life phenomena often depend on a multitude of factors and it requires more than just one variable to properly model such situations. This leads to the study of the theory of differentiation and integration of functions in several variables, called *Multivariable Calculus*. The mathematical stage on which the study of functions in several variables unfolds is the n-dimensional Euclidean space $\mathbb{R}^n$.



Before defining the n-dimensional Euclidean space and its intrinsic topology, let us recall some basic notions commonly used in analysis and calculus.

- $\mathbb{N}$ the *natural numbers* $\{1, 2, 3, 4, \ldots\}$,
- $\mathbb{Z}$ the *integers*, i.e., signed whole numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- $\mathbb{Q}$ the *rational numbers* $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$,
- $\mathbb{R}$ the *real numbers*,
- $\mathbb{C}$ the *complex numbers*,

An *open interval* is an interval that does not include its boundary points and is denoted by parentheses. The open intervals are thus one of the forms

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} : a < x < b\},\\
(-\infty,b) &= \{x \in \mathbb{R} : x < b\},\\
(a,+\infty) &= \{x \in \mathbb{R} : a < x\},\\
(-\infty,+\infty) &= \mathbb{R},
\end{aligned}$$

where a and b are real numbers with $a \leqslant b$. The interval $(a,a) = \emptyset$ is the empty set, a degenerate interval. Open intervals are *open sets* in the topology of $\mathbb{R}$.

A *closed interval* is an interval that includes all its boundary points and is denoted by square brackets. Closed intervals take the form

$$\begin{aligned}
[a,b] &= \{x \in \mathbb{R} : a \leqslant x \leqslant b\},\\
(-\infty,b] &= \{x \in \mathbb{R} : x \leqslant b\},\\
[a,+\infty) &= \{x \in \mathbb{R} : a \leqslant x\},\\
(-\infty,+\infty) &= \mathbb{R},
\end{aligned}$$

Closed intervals are *closed sets* in the topology of $\mathbb{R}$. Note that the interval $\mathbb{R} = (-\infty,+\infty)$ is both open and closed at the same time.

A *half-open interval* is a finite interval that includes one endpoint but not the other. It can be left-open or right-open, depending on which endpoint is excluded:

$$\begin{aligned}
(a,b] &= \{x \in \mathbb{R} : a < x \leqslant b\},\\
[a,b) &= \{x \in \mathbb{R} : a \leqslant x < b\},
\end{aligned}$$

Note that half-open intervals are neither open nor closed sets in the topology of $\mathbb{R}$.

Intervals of the form $[a,b]$, $[a,b)$, $(a,b]$, (a,b) for $a,b \in \mathbb{R}$ with $a \leqslant b$ are called *bounded intervals*, whereas intervals like $(-\infty,b]$, $(-\infty,b)$, $[a,+\infty)$, and $(a,+\infty)$ are *unbounded intervals*.

1.1 The vector space $\mathbb{R}^n$

Given a positive integer n, the set $\mathbb{R}^n$ is defined as the set of all ordered n-tuples $(x_1,\dots,x_n)$ of real numbers. It is called the *standard Euclidean space of dimension n*, or simply the *n-dimensional Euclidean space*.

We can represent an element of $\mathbb{R}^n$ either as an n-tuple, which is the same as a row vector with n entries,

$$\mathbf{x} = (x_1,\dots,x_n)$$

or as a column vector with n entries

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Both representations are common and widely used in the literature. We will generally use column vectors to denote elements of $\mathbb{R}^n$ in calculations, and row vectors to denote elements of $\mathbb{R}^n$ as input parameters of functions defined on $\mathbb{R}^n$.

There are also different ways in which elements in $\mathbb{R}^n$ are denoted, the three most common are

$$x, \qquad \mathbf{x}, \qquad \text{and} \qquad \vec{x}.$$

In this text, we will predominantly use x for elements in $\mathbb{R}$ and $\mathbf{x}$ for elements in $\mathbb{R}^n$ for $n \geqslant 2$.

The set $\mathbb{R}^n$ is an n-dimensional inner product vector space over the real numbers. This means it is closed under addition, scalar multiplication, and endowed with an inner product called the scalar product. The addition on $\mathbb{R}^n$ is defined coordinate wise by

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

The multiplication of an element $\mathbf{x} \in \mathbb{R}^n$ by a scalar $\lambda \in \mathbb{R}$ is defined as

$$\lambda\mathbf{x} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

The way in which addition and multiplication on $\mathbb{R}^n$ interact is described by the distributive law, which asserts that

$$\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}. \qquad \text{(Distributive Law)}$$

The vector space $\mathbb{R}^n$ is also equipped with a *scalar product* $\langle \cdot, \cdot \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k.$$

The scalar product satisfies the three following properties:

1. **Positive-definiteness:** $\langle \mathbf{x}, \mathbf{x} \rangle \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with equality only for $\mathbf{x} = \mathbf{0}$.
2. **Symmetry:** $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
3. **Bilinearity:** $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

In linear algebra, a vector $\mathbf{x}$ is also an $n \times 1$ matrix. Its transpose, written $\mathbf{x}^\top = (x_1, \ldots, x_n)$, is therefore a $1 \times n$ matrix, and we can interpret the scalar product of two vectors $\mathbf{x}, \mathbf{y}$ as the matrix product of $\mathbf{x}^\top$ and $\mathbf{y}$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = (x_1, \ldots, x_n) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

1.2 The Euclidean distance on $\mathbb{R}^n$

To be able to extend the analytical methods presented in Analysis 1 to the space $\mathbb{R}^n$, it is important to endow $\mathbb{R}^n$ with a topological structure. On $\mathbb{R}$ we have used the absolute value to define a distance $d(x, y) = |x - y|$, which was then used to define notions such as convergence and continuity in $\mathbb{R}$. We seek to generalize the absolute value and the distance to the space $\mathbb{R}^n$. To do so, we will introduce the concepts of a norm and a metric.

Definition 1.1 (The Euclidean norm on $\mathbb{R}^n$)**.** The *Euclidean norm* on $\mathbb{R}^n$ is the function $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}. \tag{1.1}$$

It measures the distance of the point $\mathbf{x}$ to the origin $\mathbf{0} = (0, \ldots, 0)$.

Observe that in one dimension, the Euclidean norm of a real number is the same as the absolute value of that number. In general, the Euclidean norm satisfies the following properties:

1. **Non-negativity:** $\|\mathbf{x}\|_2 \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with equlity if and only if $\mathbf{x} = \mathbf{0}$.
2. **Homogeneity:** $\|\lambda \cdot \mathbf{x}\|_2 = |\lambda| \cdot \|\mathbf{x}\|_2$ for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.
3. **Triangle inequality:** $\|\mathbf{x} + \mathbf{y}\|_2 \leqslant \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

One of the most important properties of the scalar product is the *Cauchy-Schwarz inequality*, which says that

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2 \tag{Cauchy-Schwarz}$$

The Euclidean norm $\|\mathbf{x}\|_2$ also corresponds to the length of a vector $\mathbf{x}$. The scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ measures the angle between the two vectors $\mathbf{x}$ and $\mathbf{y}$: if we designate θ as the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos\theta. \tag{Angle Formula}$$

In particular if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors, i.e., $\theta = \pm\pi/2$, then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. As a consequence, we obtain the famous *Pythagorean theorem*, which says that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal then

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2. \tag{Pythagoras}$$

With the help of the Euclidean norm we can define a metric on $\mathbb{R}^n$ called the Euclidean distance.

Definition 1.2 (The Euclidean distance on $\mathbb{R}^n$)**.** The *Euclidean distance* on $\mathbb{R}^n$ is the function $d(.,.) \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}. \tag{1.2}$$

The Euclidean distance captures the natural distance between two points in $\mathbb{R}^n$. It satisfies the following three properties:

1. **Non-negativity:** $d(\mathbf{x}, \mathbf{y}) \geqslant 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with equality only when $\mathbf{x} = \mathbf{y}$.
2. **Symmetry:** $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
3. **Triangle inequality:** $d(\mathbf{x}, \mathbf{y}) \leqslant d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{z})$.

1.3 The topology on $\mathbb{R}^n$

The Euclidean distance $d(\mathbf{x}, \mathbf{y})$ induces a topology on $\mathbb{R}^n$ which underpins all analytical considerations on $\mathbb{R}^n$. In particular, notions such as continuity, convergence, differentiablility and integrability are all defined in terms of this topology. The building blocks of the topology on $\mathbb{R}^n$ are the so-called open balls.

Definition 1.3 (Open Ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) < r\}$$

is called the *open ball* of radius r centered at $\mathbf{a}$.

Open balls are the mathematical conceptualization of “nearness” and an important use of open balls is to topologically distinguish distinct points: if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{y}$ then we can find a sufficiently small open ball centered at $\mathbf{x}$ and another sufficiently small open ball centered at $\mathbf{y}$ such that these two balls don’t touch.

Open balls are instances of open sets. An open set is a set with the property that if $\mathbf{x}$ is a point in the set then all points that are sufficiently near to $\mathbf{x}$ also belong to the set. The mathematically precise definition is as follows:

Definition 1.4 (Open set)**.** A subset $U \subseteq \mathbb{R}^n$ is *open* if for any point $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that the open ball $B(\mathbf{x}, \varepsilon)$ is contained in U.

The empty set $\emptyset$ and the space $\mathbb{R}^n$ are open. Also, as was already mentioned, any open ball $B(\mathbf{a}, r)$ is an open set.

Example 1.1 (Open Sets in $\mathbb{R}^n$)**.**

1. If $a < b$ are real numbers then the interval

   $$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

   is an open set. Indeed, if $x \in (a, b)$, simply take $r = \min\{x - a, b - x\}$. Both these numbers are strictly positive, since $a < x < b$, and so is their minimum. Then the “1-dimensional ball” $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$ is a subset of (a, b). This proves that (a, b) is an open set.
2. The infinite intervals (a, ∞) and $(-\infty, b)$ are also open but the intervals

   $$(a, b] = \{x \in \mathbb{R} : a < x \leqslant b\} \qquad \text{and} \qquad [a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

   are not open sets.
3. The rectangle

   $$(a, b) \times (c, d) = \{(x, y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\}$$

is an open set.

The antithetical notion to an open set is that of a closed set.

Definition 1.5 (Closed set)**.** A subset $C \subseteq \mathbb{R}^n$ is *closed* if its complement $\mathbb{R}^n \backslash C$ is open.

The empty set $\emptyset$ and the space $\mathbb{R}^n$ are the only sets that are both closed and open at the same time. Intuitively, one should think of a closed set as a set that has no "punctures" or "missing endpoints", i.e., it includes all limiting values of points. For instance, the punctured plane $\mathbb{R}^2 \backslash \{(0,0)\}$ is not a closed set.

An example of a closed set is the closed ball.

Definition 1.6 (Closed Ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$\overline{B(\mathbf{a}, r)} = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) \leqslant r\}$$

is called the closed ball of radius r centered at $\mathbf{a}$. It is a closed set.

Example 1.2 (Closed Sets in $\mathbb{R}^n$)**.**

1. The closed interval

$$[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

is a closed set, because its complement $\mathbb{R} \backslash [a, b] = (-\infty, a) \cup (b, \infty)$ is an open set.
2. Infinite intervals with closed boundary $[a, \infty)$ and $(-\infty, b]$ are closed sets.
3. Halfopen intervals such as $[a, b)$ or $(a, b]$ are neither closed nor open sets.
4. Any set consisting of only finitely many points is a closed set.

The following two propositions describe how open and closed sets behave under basic set manipulations such as unions, intersections, and set differences.

Proposition 1.1.

- *If $U \subseteq \mathbb{R}^n$ is open and $C \subseteq \mathbb{R}^n$ is closed then $U \backslash C$ is open.*
- *If $C \subseteq \mathbb{R}^n$ is closed and $U \subseteq \mathbb{R}^n$ is open then $C \backslash U$ is closed.*

Proposition 1.2.

- *If $U_1, \ldots, U_k \subseteq \mathbb{R}^n$ are open then $U_1 \cup \ldots \cup U_k$ and $U_1 \cap \ldots \cap U_k$ are open.*
- *If $C_1, \ldots, C_k \subseteq \mathbb{R}^n$ are closed then $C_1 \cup \ldots \cup C_k$ and $C_1 \cap \ldots \cap C_k$ are closed.*

To better grasp the difference between open sets and closed sets, we introduce the concept of interior points, exterior points, and boundary points.

Definition 1.7 (Interior, Exterior, Boundary Points)**.** Let S be a subset of $\mathbb{R}^n$ and $\mathbf{x}$ a point in $\mathbb{R}^n$.

(i) We call $\mathbf{x}$ an *interior point* of S if there exists $r > 0$ such that the ball $B(\mathbf{x}, r)$ is contained in S.

(ii) We call $\mathbf{x}$ an *exterior point* of S if there exists $r > 0$ such that the ball $B(\mathbf{x}, r)$ has empty intersection with S.

(iii) We call $\mathbf{x}$ a *boundary point* of S if it is neither an interior point nor an exterior point for S. Equivalently, $\mathbf{x}$ is a boundary point of S if for every $r > 0$ the ball

$B(\mathbf{x}, r)$ has non-empty intersection with S without being entirely contained in S.

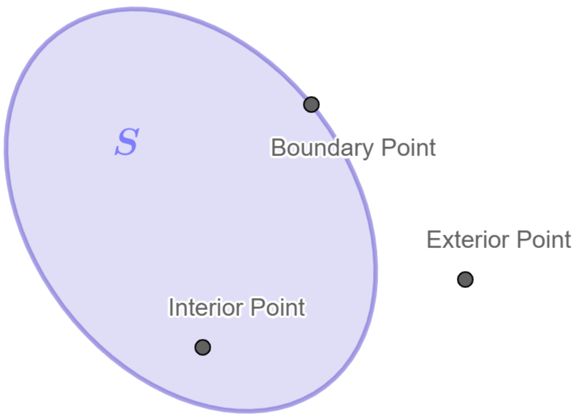


Figure 1.1: Illustration of the difference between interior, exterior and boundary points of a set S.

Note that every point is either interior, exterior or on the boundary in relationship to a set S.

Definition 1.8 (Interior)**.** The set of all interior points of a set S is called the interior of S and it is denoted by $\mathring{S}$.

Definition 1.9 (Boundary)**.** The set of all boundary points of a set S is called the boundary of S and we use ∂S to denote it.

Definition 1.10 (Closure)**.** The closure of S, denoted by $\overline{S}$, is the set of points $\mathbf{x} \in \mathbb{R}^n$ with the property that for all $r > 0$ one has

$$B(\mathbf{x}, r) \cap S \neq \emptyset.$$

Equivalently, the closure of S is the union of all its interior points and all its boundary points.

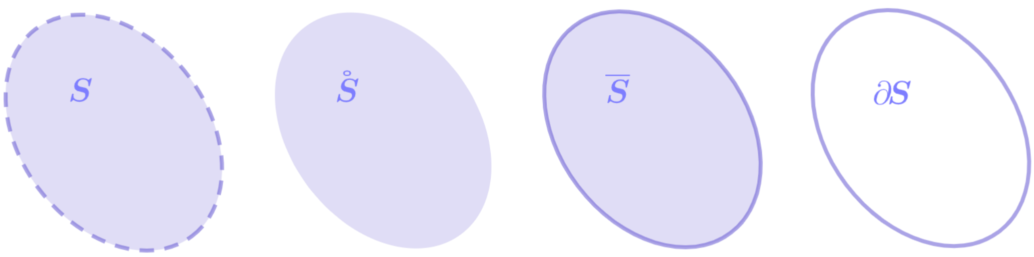


Figure 1.2: The interior, closure and boundary sets of a set S.

Clearly, we have the set inclusions $\mathring{S} \subseteq S \subseteq \overline{S}$. To summarize, the closure of S is S plus its boundary, its interior is S minus its boundary, and the boundary is the closure minus the interior:

$$\mathring{S} = S \backslash \partial S \qquad \overline{S} = S \cup \partial S, \qquad \text{and} \qquad \partial S = \overline{S} \backslash \mathring{S}.$$

Proposition 1.3. *Let $S \subseteq \mathbb{R}^n$. The interior $\mathring{S}$ is the largest open set contained inside of S. The closure $\overline{S}$ is the smallest closed set that has S as a subset.*

Corollary 1.1. *A set is open if and only if it is equal to its interior. On the other hand, a set is closed if and only if it is equal to its closure, which is the same as saying that it contains all its boundary points.*

Example 1.3 (Closure, Interior, Boundary)**.**

1. The sets $(0,1)$, $[0,1]$, $[0,1)$, and $(0,1]$ all have the same closure, interior, and boundary: the closure is $[0,1]$, the interior is $(0,1)$, and the boundary consists of the two points 0 and 1.
2. The sets

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \qquad \text{and} \qquad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$$

   both have the same closure, interior, and boundary: the closure is the disc of equation $x^2 + y^2 \leqslant 1$, the interior is the disc of equation $x^2 + y^2 < 1$, and the boundary is the circle of equation $x^2 + y^2 = 1$.
3. The set

$$U = \{(x,y) \in \mathbb{R}^2 : |y| < x^2\}$$

   describes the region between two parabolas touching at the origin, shown in Fig. 1.3. The set is open, so $U = \mathring{U}$. The closure of U is given by

$$\overline{U} = \{(x,y) \in \mathbb{R}^2 : |y| \leqslant x^2\}.$$

   In particular, the closure contains the point $(0,0)$.

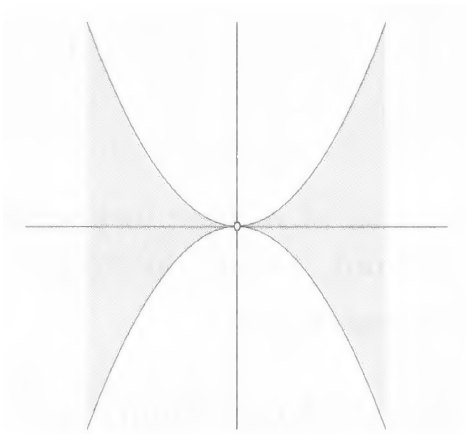


Figure 1.3: The origin belongs to the closure of the shaded region.

4. The unit ball is open in $\mathbb{R}^n$ and is defined by
$$B_1 = B(\mathbf{0}, 1) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$$
Its boundary is the sphere $\partial B_1 = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$.
5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. The set
$$G_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$
is known as the graph of f and represents a curve in $\mathbb{R}^2$. We have $\mathring{G}_f = \emptyset$. Therefore $G_f = \partial G_f$. The closed graph theorem says that graph $\mathring{G}_f$ is a closed set in $\mathbb{R}^2$ if f is a continuous function.
6. Let $B = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1\}$ and $I = [0, 5]$. The set S defined by
$$S = B \times I = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } 0 \leqslant x_3 \leqslant 5\right\}$$
is a cylinder. The set S is neither closed nor open. The boundary of S is given by
$$\partial S = \underbrace{\partial B \times I}_{E_1} \cup \underbrace{B \times \partial I}_{E_2},$$
where
$$E_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1 \text{ and } 0 \leqslant x_3 \leqslant 5\right\},$$
$$E_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } x_3 \in \{0, 5\}\right\}.$$

Definition 1.11 (Neighborhood of a point in $\mathbb{R}^n$)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$. If $\mathbf{x}$ is an interior point of U then U is called a *neighborhood of* $\mathbf{x}$.

1.4 Sequences in $\mathbb{R}^n$

Limits of sequences and limits of functions are fundamental notions in calculus, as you already have seen in Analysis 1. Let us extend these principles to higher dimensions. We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of natural numbers.

Definition 1.12 (Sequences in $\mathbb{R}^n$)**.** A *sequence* of elements of $\mathbb{R}^n$ is a function $k \mapsto \mathbf{x}_k$ that associates to every natural number $k \in \mathbb{N}$ an element $\mathbf{x}_k \in \mathbb{R}^n$. We write $(\mathbf{x}_k)_{k\in\mathbb{N}}$ to denote a sequence in $\mathbb{R}^n$.

Although $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is by definition a sequence of n-tuples, we can also think of it as an n-tuple of sequences by considering each coordinate as an individual sequence,
$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix}.$$

Definition 1.13 (Convergent sequence)**.** A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{x} \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists $N > 1$ such that when $k \geqslant N$, then

$d(\mathbf{x}_k, \mathbf{x}) < \varepsilon$. In this case we call $\mathbf{x}$ the *limit* of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and write

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}.$$

Note that not every sequence has a limit, but if a sequence does then this limit is unique. Sequences that possess a limit are called *convergent*, whereas sequences that don't possess one are called *divergent*.

It follows from Definition 1.13 that a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ if and only if the sequence of distances $d(\mathbf{x}_k, \mathbf{x})$ converges to 0, i.e.,

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \quad \Longleftrightarrow \quad \lim_{k\to+\infty} d(\mathbf{x}_k, \mathbf{x}) = 0.$$

Convergence is also observed coordinate wise: A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ if and only if each coordinate of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to the respective coordinate of $\mathbf{x}$. More precisely, if

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \quad \Longleftrightarrow \quad \lim_{k\to+\infty} x_{i,k} = x_i \ \text{ for all } i = 1, \ldots, n.$$

Example 1.4 (Convergent and divergent sequences in $\mathbb{R}^n$)**.**

1. The sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{x}_k = \begin{pmatrix} e^{-k} \\ \frac{k}{k+1} \\ \frac{1}{\sqrt{k^2-k}-k} \end{pmatrix}$$

   converges as $k \to +\infty$ to the limit

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},$$

   because $\lim_{k\to+\infty} e^{-k} = 0$, $\lim_{k\to+\infty} \frac{k}{k+1} = 1$, and $\lim_{k\to+\infty} \frac{1}{\sqrt{k^2-k}-k} = -2$.

2. The sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{x}_k = \begin{pmatrix} 0 \\ \frac{1-(-1)^k}{2} \end{pmatrix}$$

   diverges because it diverges in the second coordinate.

The following properties describe the arithmetic operations of sequences in the n-dimensional Euclidean space and tell us that limits cooperate nicely with the vector space structure of $\mathbb{R}^n$. Let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}^n$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

1. **Addition of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ both converge then so does $(\mathbf{x}_k + \mathbf{y}_k)_{k\in\mathbb{N}}$ and
$$\lim_{k\to+\infty} \mathbf{x}_k + \mathbf{y}_k = \lim_{k\to+\infty} \mathbf{x}_k + \lim_{k\to+\infty} \mathbf{y}_k.$$
2. **Multiplication of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ both converge then so does $(\lambda_k\mathbf{x}_k)_{k\in\mathbb{N}}$ and
$$\lim_{k\to+\infty} \lambda_k\mathbf{x}_k = \left(\lim_{k\to+\infty} \lambda_k\right) \cdot \left(\lim_{k\to+\infty} \mathbf{x}_k\right).$$
3. **Inner product of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ both converge then so does $(\langle \mathbf{x}_k,\, \mathbf{y}_k\rangle)_{k\in\mathbb{N}}$ and
$$\lim_{k\to+\infty} \langle \mathbf{x}_k,\, \mathbf{y}_k\rangle = \left\langle \lim_{k\to+\infty} \mathbf{x}_k,\, \lim_{k\to+\infty} \mathbf{y}_k \right\rangle.$$

Definition 1.14 (Cauchy sequences)**.** A sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N > 1$ such that $k, l \geqslant N$ implies $d(\mathbf{x}_k, \mathbf{x}_l) < \varepsilon$.

Theorem 1.1. *Every convergent sequence* $(\mathrm{x}_k)_{k\in\mathbb{N}}$ *is a Cauchy sequence and every Cauchy sequence is convergent.*

Proposition 1.4. *Let* $S \subseteq \mathbb{R}^n$ *be a non-empty set and suppose* $\mathbf{x} \in \partial S$ *is a boundary point of* S*. Then there exists a sequence of elements in* $\mathring{S}$, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \in \mathring{S}$*, such that*
$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}.$$

The following example provides an illustration of the content of Proposition 1.4.

Example 1.5. Consider the open ball of radius 5 centered at the origin in $\mathbb{R}^2$,
$$B(\mathbf{0}, 5) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 5\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 25\}.$$
The boundary of $B((0, 0), 5)$ is the circle of radius 5 centered at the origin, i.e.,
$$\partial B(\mathbf{0}, 5) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 5\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 25\}.$$
Any point $\mathbf{x} \in \partial B(\mathbf{0}, 5)$ of this circle takes the form
$$\mathbf{x} = \begin{pmatrix} 5\cos\theta \\ 5\sin\theta \end{pmatrix}, \qquad \text{for some } \theta \in [0, 2\pi).$$
We can define a sequence
$$\mathbf{x}_k = \begin{pmatrix} \frac{5k}{k+1}\cos\theta \\ \frac{5k}{k+1}\sin\theta \end{pmatrix},$$
and note that $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}$. So we see that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence of points inside the open ball $B(\mathbf{0}, 5)$ converging to the point $\mathbf{x}$ on the border .

Proposition 1.5. *Let* $S \subseteq \mathbb{R}^n$ *be a non-empty subset of* $\mathbb{R}^n$ *and let* $(\mathbf{x}_k)_{k\in\mathbb{N}}$ *be a sequence of elements in* S*. If* $(\mathbf{x}_k)_{k\in\mathbb{N}}$ *converges then the limit* $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}$ *must* *belong to $\overline{S}$, the closure of S.*

Example 1.6. Consider the "halfopen" rectangle

$$S = [0,1] \times [0,1).$$

This is not a closed set, because the point $(\frac{2}{3}, 1)$, for example, is in the boundary ∂S but not in S itself. Moreover, the sequence

$$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{3}{4} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \dots$$

is a sequence of points in the interior of S that converge to the point $(\frac{2}{3}, 1)$, which is not part of S, but it is part of the closure of S.

Definition 1.15 (Bounded set)**.** A subset $E \subseteq \mathbb{R}^n$ is *bounded* if it is contained in a ball of finite radius centered at the origin:

$$E \subseteq B(\mathbf{0}, R) \quad \text{for some } R < \infty.$$

Note that a closed set need not be bounded. For instance, the interval $[0, \infty)$ is closed, but it is not a bounded.

Definition 1.16 (Compact set)**.** A subset $C \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

Compactness is the basic "finiteness criterion" for subsets of $\mathbb{R}^n$. An important characterization of compact sets in Euclidean spaces is given by the Bolzano-Weierstrass theorem. Before we can state this theorem, we need to recall what is a subsequence.

Definition 1.17 (Subsequence)**.** A *subsequence* of a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is any sequence of the form $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$, where $(k_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of positive integers.

If a sequence converges then any subsequence of it also converges to the same limit.

Theorem 1.2 (Bolzano-Weierstrass theorem in $\mathbb{R}^n$)**.** *Let $C \subseteq \mathbb{R}^n$ be compact. Any sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of elements in C possesses a convergent subsequence $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$ whose limit is in C.*

Definition 1.18 (Bounded sequences in $\mathbb{R}^n$)**.** A sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is *bounded* if there exists a constant $C > 0$ such that $\|\mathbf{x}_k\|_2 \leqslant C$ for any $k \in \mathbb{N}$.

Note that every convergent sequence is a bounded sequence, but the opposite is in general not true. For example, the sequence $x_k = (-1)^k$ is bounded and does not converge. The following is an immediate corollary of the Bolzano-Weierstrass theorem.

Corollary 1.2. *Each bounded sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ has a convergent subsequence $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$.*